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Three Classes of Orthomodular Lattices[3]

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Let $\mathcal{OML}$ denote the class of all orthomodular lattices and $\mathcal{C}$ denote the class of those that are commutator-finite. Also, let $\mathcal{C}_1$ denote the class of orthomodular lattices that satisfy the block extension property, $\mathcal{C}_2$ those that satisfy the weak block extension property, and $\mathcal{C}_3$ those that are locally finite. We show that the following strict containments hold: $\mathcal{C} \subset \mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}_3 \subset \mathcal{OML}$.

KEY WORDS: orthomodular lattice (OML); commutator-finite OML; block-finite OML; locally finite OML; block extension property; weak block extension property; Greechie space; Loop Lemma; orthogonality space.

1. INTRODUCTION

In the concluding remarks of Bruns and Greechie (1990) and in Legan (1998), the authors discuss several classes of orthomodular lattices. Continuing that discussion, we adopt the following notation. Let us denote the class of all orthomodular lattices by $\mathcal{OML}$ and the class of commutator-finite orthomodular lattices by $\mathcal{C}$. Additionally, we denote the class of orthomodular lattices that satisfy the block extension property by $\mathcal{C}_1$, the class of those that satisfy the weak block extension property by $\mathcal{C}_2$, and the class of those that are locally finite by $\mathcal{C}_3$. After making the appropriate definitions, we give two examples and refer to two others to show that these classes are ordered by strict containment ($\subset$); namely, $\mathcal{C} \subset \mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}_3 \subset \mathcal{OML}$.

An *orthomodular lattice* (abbreviated, *OML*) is an ortholattice L satisfying the condition that if $a \leq b$ and $b \wedge a' = 0$, then $a = b$. A *commutator* in an OML L is an element of the form $(a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b')$. A maximal boolean subalgebra of an OML L is called a *block* of L. An OML is *commutator-finite* (respectively, *block-finite*), if it contains only finitely many commutators

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(respectively, blocks). The set of all blocks of L is denoted by $\mathfrak{A}_L$. A subalgebra M of an OML L is said to be *full* when $\mathfrak{A}_M \subseteq \mathfrak{A}_L$.

We say that an OML L has the *block extension property* (abbreviated, *BEP*) if, for every finite set $\mathfrak{B}$ of blocks of L, there exists a block-finite full subalgebra S of L containing $\bigcup \mathfrak{B}$. Also, we say that an OML L has the *weak block extension property* (abbreviated, *WBEP*) if, for every finite set $\mathfrak{B}$ of blocks of L, the subalgebra, $\Gamma(\bigcup \mathfrak{B})$, of L generated by the elements of $\bigcup \mathfrak{B}$ is block-finite. Finally, we call an OML L *locally finite* if every finite set of elements of L generates a finite subalgebra of L. Our discussion is always within an orthomodular lattice. We assume a fair amount of familiarity with the basic notation, terminology, and results of orthomodular lattice theory, and refer the reader to the bibliography for details.

In order to prove that a commutator-finite OML L is locally finite, Bruns and Greechie first show that every commutator-finite OML L has the block-extension property. Consequently, each finite generating set in L, since it is contained in the union of a finite set of blocks of L, is contained in a full block-finite subalgebra of L. Hence, by the main result of Bruns (1978), which states that block-finiteness implies local-finiteness, each such set generates a finite subalgebra of L. In other words, commutator-finiteness implies local-finiteness. So, the BEP plays a crucial role in this result. It is clear that every OML that has the BEP also has the WBEP since subalgebras of block-finite OMLs are again block-finite. Moreover, the WBEP also implies local finiteness in the same way as does the BEP.

In the concluding remarks of Bruns and Greechie (1990), the authors pose several questions. The question we answer here is whether or not the BEP is strictly stronger than the WBEP. The example given here shows that it is. We also give an example to show that the class of OMLs possessing the WBEP is strictly contained in the class of locally finite OMLs.

2. AN OML POSSESSING THE WBEP BUT NOT THE BEP

In Fig. 1, we have an example of an OML L_1 which possesses the WBEP, but not the BEP. The pattern is repetitive; here is the listing of the atoms in each block. They are, for each $i \in \mathbb{N}$, $\{a_{ij}, b_{ij}, a_{i,j+1}\}$, $j = 1, 2, 3, 4, 5$, and $\{a_{i6}, b_{i6}, a_{i1}\}$, comprising the infinitely many hexagons, and $\{a_{i1}, c_{i1}, a_{i+1,2}\}$, $\{a_{i2}, c_{i2}, d_i, a_{i+1,4}\}$, $\{a_{i3}, c_{i3}, a_{i+1,6}\}$, $\{a_{i4}, c_{i4}, a_{i+1,2}\}$, $\{a_{i5}, c_{i5}, d_i, a_{i+1,4}\}$, and $\{a_{i6}, c_{i6}, a_{i+1,6}\}$, comprising the atoms of the blocks between the ith and the $(i+1)^{\text{st}}$ hexagon immediately to its right. Each block in any hexagon and all others except the two that contain d_i, for all $i \in \mathbb{N}$, are copies of $\mathbf{2^3}$. The two that contain d_i, for all $i \in \mathbb{N}$, are copies of $\mathbf{2^4}$. The drawing completely describes the OML L_1.

Firstly, we must show that L_1 is an OML. We observe that it is not a Greechie space by considering the atoms d_i and $a_{i+1,4}$, which both appear in the same

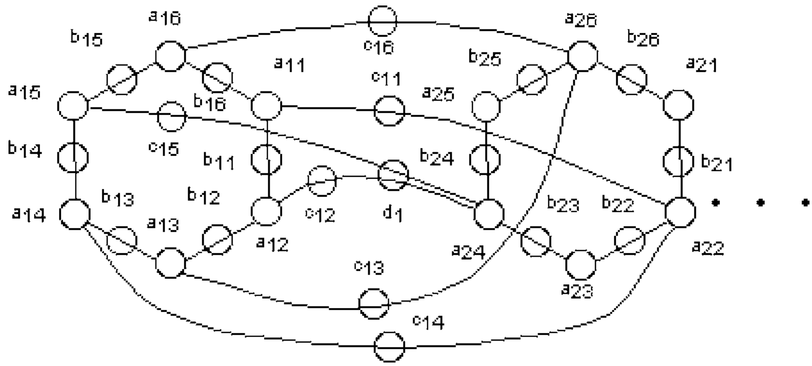

Fig. 1. L_1, an example having the WBEP but not the BEP.

two blocks, one having as its set of atoms $\{a_{i2}, c_{i2}, d_i, a_{i+1,4}\}$ and the other $\{a_{i5}, c_{i5}, d_i, a_{i+1,4}\}$, for all $i \in \mathbb{N}$. However, we can prove that L_1 is an OML by using orthogonality space techniques which we now present.

To do this, we first make the following definitions. An ordered pair $(X, \perp)$ is called an *orthogonality space* if X is a non-empty set and $\perp \subseteq X \times X$ is an anti-reflexive, symmetric binary relation on X; that is, for all $x, y \in X$, $x \perp x$ fails, and $x \perp y$ implies that $y \perp x$.

Let $(X, \perp)$ be an orthogonality space. If $M \subseteq X$, then we define $M^{\perp}$, called the *orthogonal* of M, by the equation $M^{\perp} := \{x \in X | x \perp m$ for every $m \in M\}$. $D \subseteq X$ is called a $\perp$*-(sub)set* in case $d_1 \perp d_2$ holds whenever $d_1, d_2 \in D$ with $d_1 \neq d_2$.

If $M \subseteq X$, we say that M is *orthogonally-closed* (abbreviated, $\perp$*-closed*) in case $M = M^{\perp\perp}$. We define $\mathcal{P}^{\perp}(X)$ by the equation $\mathcal{P}^{\perp}(X) := \{M \in \mathcal{P}(X) | M$ is $\perp$-closed$\}$.

Let *atL* denote the set of atoms of L. Let $X = atL_1$. These are represented by the vertices in Fig. 1. For $x, y \in X, x \perp y$ if and only if $x \neq y$ and both x and y are on a common line in Fig. 1. The maximal $\perp$-sets determine the lines in the Greechie diagram of L_1: these are in one-to-one correspondence with the blocks of L_1.

The following results then hold. The poset $(\mathcal{P}^{\perp}(X), \subseteq)$ is a complete ortholattice under the orthocomplementation $M \mapsto M^{\perp}$; and $\mathcal{P}^{\perp}(X)$ is an OML if and only if, for every $M \in \mathcal{P}^{\perp}(X)$ and for every maximal $\perp$-subset D of M, $D^{\perp\perp} = M$ holds. (See Kalmbach, 1983, page 262.)

We now use these results to show that $\mathcal{P}^{\perp}(X)$ is an OML. To accomplish this, we give an exhaustive accounting of the $\perp$-closed subsets $M \subseteq X$ and their maximal $\perp$-subsets ("bases") $D \subseteq M$; the raeder is asked to verify that for all such inclusions in Table I, where, for each of these, $i \in \mathbb{N}$ and $a \in$ at L_1, we have $D^{++} = M$. The meaning should remain clear in view of the preceding paragraphs.

Table I. Orthogonally Closed Subsets and Their "Bases"

$M = M^{\perp\perp}$	$D \subseteq M$
$\emptyset$	$\emptyset$
$\emptyset^{\perp} = X$	The atoms in just one of any of the blocks
$\{a\}$	$\{a\}$
$\{a\}^{\perp}$	The atoms in just one of any of the blocks which contain a, except for a itself
$\{d_i, a_{i+1,4}\}$	$\{d_i, a_{i+1,4}\}$
$\{d_i, a_{i+1,4}\}^{\perp} = \{a_{i2}, c_{i2}, a_{i5}, c_{i5}\}$	$\{a_{i2}, c_{i2}\}, \{a_{i5}, c_{i5}\}$
$\{a_{i2}, a_{i+1,4}\}$	$\{a_{i2}, a_{i+1,4}\}$
$\{a_{i2}, a_{i+1,4}\}^{\perp} = \{c_{i2}, d_i\}$	$\{c_{i2}, d_i\}$
$\{a_{i2}, d_i\}$	$\{a_{i2}, d_i\}$
$\{a_{i2}, d_i\}^{\perp} = \{c_{i2}, a_{i+1,4}\}$	$\{c_{i2}, a_{i+1,4}\}$
$\{a_{i5}, a_{i+1,4}\}$	$\{a_{i5}, a_{i+1,4}\}$
$\{a_{i5}, a_{i+1,4}\}^{\perp} = \{c_{i5}, d_i\}$	$\{c_{i5}, d_i\}$
$\{a_{i5}, d_i\}$	$\{a_{i5}, d_i\}$
$\{a_{i5}, d_i\}^{\perp} = \{c_{i5}, a_{i+1,4}\}$	$\{c_{i5}, a_{i+1,4}\}$

We now define a sequence of subalgebras which are useful for showing that the WBEP holds, but the BEP does not. For $i \in \mathbb{N}$, define e_i to be $d_i \vee a_{i+1,4}$; and define $M_i \leq L_1$ to be the union of all blocks to the left of the ith hexagon, the ith hexagon, $\{a_{i1}, c_{i1}, a_{i+1,2}\}$, $\{a_{i3}, c_{i3}, a_{i+1,6}\}$, $\{a_{i4}, c_{i4}, a_{i+1,2}\}$, $\{a_{i6}, c_{i6}, a_{i+1,6}\}$, $\{a_{i2}, c_{i2}, e_i\}$, $\{a_{i5}, c_{i5}, e_i\}$, $\{a_{i+1,1}, b_{i+1,1}, a_{i+1,2}\}$, and $\{a_{i+1,6}, b_{i+1,6}, a_{i+1,1}\}$. Then each M_i is a block-finite subalgebra of L_1 which is not full. In Fig. 2, we show the Greechie diagram of M_1.

To show that L_1 has the WBEP, let $\mathcal{B}$ be any collection of finitely many blocks of L_1. Pick $j \in \mathbb{N}$ so that all of the blocks in $\mathcal{B}$ are to the left of the jth hexagon.

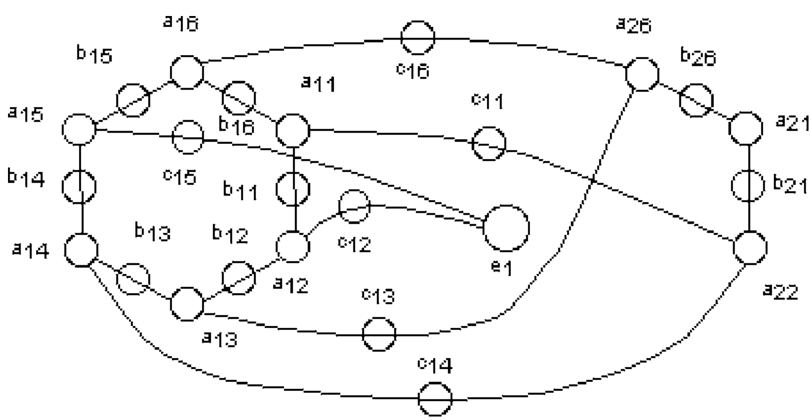

Fig. 2. M_1, a block-finite subalgebra of L_1 which is not full.

Since $\bigcup \mathcal{B} \subseteq M_j \leq L_1$ and subalgebras of block-finite OMLs are block-finite, it follows that $\Gamma(\bigcup \mathcal{B})$ is block-finite.

To verify the failure of the BEP, choose $\mathcal{B} := \{\{a_{12}, b_{12}, a_{13}\}, \{a_{15}, b_{15}, a_{16}\}\}$ and observe that $\Gamma(\bigcup \mathcal{B}) = M_1$. Observe that any full subalgebra of L_1 that contains $\bigcup \mathcal{B}$ must, therefore, contain $\{a_{12}, c_{12}, d_1, a_{24}\}$ and $\{a_{15}, c_{15}, d_1, a_{24}\}$. However, $\Gamma((\bigcup \mathcal{B}) \cup \{a_{12}, c_{12}, d_1, a_{24}\} \cup \{a_{15}, c_{15}, d_1, a_{24}\}) = M_2$. In a similar way, if any full subalgebra contains M_i, then it also contains M_{i+1}, for all $i \geq 2$. Hence, the only full subalgebra containing this finite choice of blocks is L_1 itself. Since L_1 is not block-finite, it does not have the BEP.

3. A LOCALLY FINITE OML NOT POSSESSING THE WBEP

In Fig. 3, we present an example of an OML L_2 which is locally finite but does not possess the WBEP. Observe also that L_2 possesses a collection of four infinite blocks, $\mathcal{B} := \{A, B, C, D\}$. The blocks of L_2 have as sets of atoms $\{a_{ij}, b_{ij}, a_{i,j+1}\}$, $j = 1, 2, 3, 4, 5$, and $\{a_{i6}, b_{i6}, a_{i1}\}$, where $i \in \mathbb{N}$, comprising the infinitely many hexagons, and $at A = \{b_{13}, b_{16}, b_{25}, b_{22}, b_{33}, b_{36}, b_{45}, b_{42}, \ldots\}$, $at B = \{b_{11}, b_{21}, b_{31}, \ldots\}$, $at C = \{b_{14}, b_{24}, b_{34}, \ldots\}$, and $at D = \{b_{15}, b_{12}, b_{23}, b_{26}, b_{35}, b_{32}, b_{43}, b_{46}, \ldots\}$; these comprise the four infinite lines each of which intersects each hexagon in one (for B and C) or two (for A and D) atoms. Each block in any hexagon is a copy of $\mathbf{2}^3$, and each of the four infinite blocks is a copy of the power set of the integers, $\mathcal{P}(\mathbb{Z})$.

The second example, L_2, is a Greechie space since any two lines in its Greechie diagram intersect in at most one point. Recall that a *loop of order n* in a Greechie space is a sequence of n lines where each line intersects the successive line in one point, non-successive lines do not intersect, and the last line intersects the first line in one point also. Note that L_2 satisfies the Loop Lemma condition for being an OML, which simply requires that there are no loops of order less than five.

The drawing completely describes the OML L_2. Moreover, L_2 is locally finite and does not have the WBEP for the following reasons.

Firstly, for any finite collection K of elements of L_2, there exists $n \in \mathbb{N}$ such that only the four infinite blocks and the first n hexagons contain elements of K. However, an atom from a hexagon further to the right than the nth hexagon may be generated by elements that are only in the same infinite block. Now, each infinite block of L_2 contains finitely many elements of K, and blocks are locally finite. Moreover, two elements from two different infinite blocks together do not generate a new element, except possibly if they are also from the same hexagon. Also, an element x from a given hexagon and another element y from an infinite block to which x does not also belong do not generate a new element, except possibly an element in the given hexagon. So, there exists $m \in \mathbb{N}$ with $m \geq n$ such

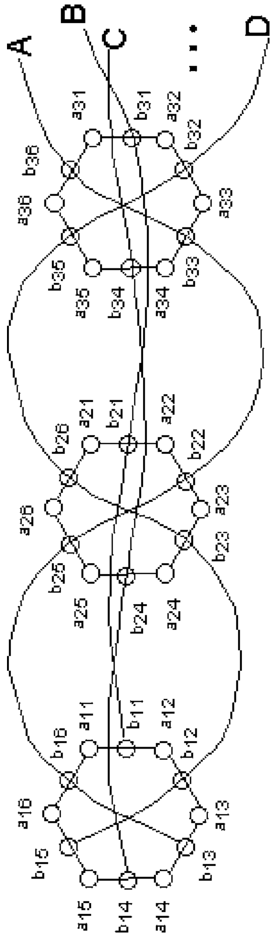

Fig. 3. L_2, an example of a locally finite OML that does not possess the WBEP.

that the subalgebra generated by K, $\Gamma(K)$, includes elements from only the first m hexagons and finite subsets of the four infinite blocks. So, $\Gamma(K)$ is finite. Hence, L_2 is locally finite.

Secondly, $\mathcal{B} := \{A, B, C, D\}$ is a finite collection of blocks whose union generates L_2 itself, which has infinitely many blocks. Therefore, L_2 does not have the WBEP.

It is, of course, true that the horizontal sum of infinitely many copies of OMLs possessing the BEP and having a commutator poset which is not contained in $\{0, 1\}$ possesses the BEP but is not commutator-finite. In particular, the horizontal sum of infinitely many copies of G_{12} (the 12-element OML having two 8-element blocks) is the simplest example of such an OML. Let us denote this example by N_1. Furthermore, let us denote the example from Greechie (1977) (which may also be found in Harding, 2002) by N_2. From our Introduction, we know that $\mathcal{C} \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3 \subseteq \mathcal{OML}$. However, the existence of N_1 allows us to conclude that $\mathcal{C}_1 \nsubseteq \mathcal{C}$, of L_1 that $\mathcal{C}_2 \nsubseteq \mathcal{C}_1$, of L_2 that $\mathcal{C}_3 \nsubseteq \mathcal{C}_2$, and of N_2 that $\mathcal{OML} \nsubseteq \mathcal{C}_3$. Therefore, $\mathcal{C} \subset \mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}_3 \subset \mathcal{OML}$.

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